Renormalization of an effective model Hamiltonian by a counter term

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An ill-defined integral equation for modeling the mass-spectrum of mesons is regulated with an additional but unphysical parameter. This parameter dependance is removed by renormalization. Illustrative graphical examples are given.

We focus on the integral equation

$$\left[M^2 - 4m^2 - 4\vec{k}^2\right]\phi(\vec{k})=\int d^3\vec{k}'\; U(\vec{k}',\vec{k})\phi(\vec{k}'),$$

with the attractive kernel

$$U(\vec{k}',\vec{k}) = -\frac{4}{3\pi^2}\frac{\alpha}{m}\left[\frac{2m^2}{(\vec{k}'-\vec{k})^2}+1\right].$$

It has two parameters α, m.

From a physical point of view the equation is a QCD-inspired effective one-body bound-state equation for modeling mesons with different constituent quark flavors [1]. M^2 are the invariant mass squares of the physical mesons, while $m = m_1 = m_2$ is the effective mass of the quark and anti-quark. It takes this explicit form due to an over-simplification by the $\uparrow\downarrow$-model [1]. If one Fourier tranforms the kernel U to configuration space, the interaction potential consists of a long-ranged Coulomb-interaction and a short-ranged delta-interaction. It is this latter part, which generates all the well known trouble. In order to get reasonable solutions one has to regulate the high momentum transfers $Q^2 = (\vec{k}'-\vec{k})^2$. Therefore we substitute the number 1 by a regulating function, $1 \to R(\Lambda, Q)$, for which the soft cut-off

$$R(\Lambda, Q^2) = \frac{\Lambda^2}{\Lambda^2+Q^2} = \frac{\Lambda^2}{\Lambda^2+(\vec{k}'-\vec{k})^2}$$

is chosen. In configuration space the short-ranged delta is now smeared out to a Yukawa interaction. Since the regulator Λ is an additional but unphysical parameter, one has to renormalize the equation in order to restore the original problem in the limit $\Lambda \to \infty$. For getting a greater tranparency we want to interpret the physical parameters α and m as renormalization constants.

That we are dealing here with a bound-state equation on the light-cone, can not be seen explicitely. The above equation results from a variable transform in the longitudinal momentum fraction x. For equal masses the relation is given by [1]

$$x(k_z) = \frac{1}{2}\left(1+\frac{k_z}{\sqrt{m^2+\vec{k}_\perp^2+k_z^2}}\right).$$

The relationship between the light-cone wavefunction ψ and the function ϕ is given by

$$\psi(x,\vec{k}_\perp) = \frac{\phi(\vec{k})}{\sqrt{x(1-x)}}\left[1+\frac{\vec{k}^2}{m^2}\right]^{\frac{1}{4}}.$$

The function ϕ has no physical meaning in the sense of a probalility amplitude and is refered to as the reduced wavefunction.

After regularization one faces an integral equation with three parameters α, m and Λ. For simplification the functions ϕ are restricted to the calculation of s-waves: $\phi(\vec{k}) = \phi(|\vec{k}|)$ and by reasons explained below, we fix $m = 406$ MeV.

The spectrum of the bound-state mass squares $M_i^2(\alpha,\Lambda)$ are then calculated numerically. For

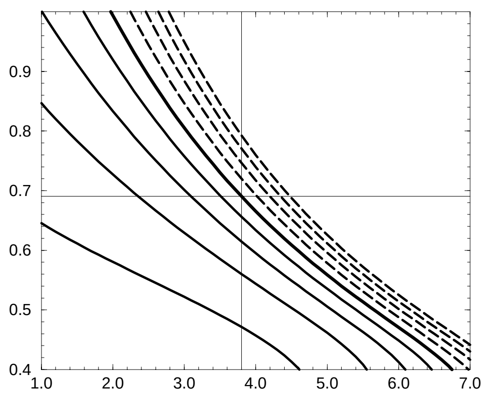

Figure 1. Nine contours $0.4 \leq \alpha_n(\Lambda) \leq 1.0$ are plotted versus $1.0 \leq \Lambda/\Delta \leq 7.0$ from bottom to top with $n = 4, 3, \cdots, -3, -4$. The contours are obtained by $M_0^2(\alpha, \Lambda) = n\Delta^2 + M_\pi^2$. The thick contour $n = 0$ describes the pion with $M_0^2 = M_\pi^2$. Masses are given in units of $\Delta = 350$ MeV.

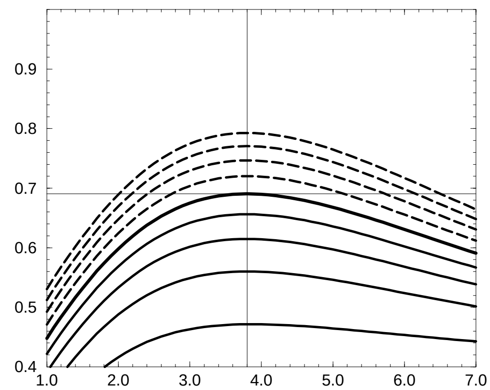

Figure 2. Nine contours $0.4 \leq \alpha_n(\Lambda) \leq 1.0$ are plotted versus $1.0 \leq \Lambda/\Delta \leq 7.0$ from bottom to top with $n = 4, 3, \cdots, -3, -4$. Here the partially renormalized $M_0^2(\Lambda, \alpha) = n\Delta^2 + M_\pi^2$ with $\Delta = 350$ MeV is displayed by contours. The thick contour $n = 0$ describes the pion with $M_0^2 = M_\pi^2$.

the ground state $M_0^2(\alpha, \Lambda)$ this is displayed in Figure 1. A similar graph could be given for the first excited state $M_1^2(\alpha, \Lambda)$.

1. Example for local renormalization

The new parameter Λ appears due to regularization. According to renormalization theory the spectrum may not depend on this formal parameter, thus we must require

$$\delta_\Lambda M^2(\Lambda) \stackrel{!}{=} 0. \tag{1}$$

To achieve this, we extend the model interaction by a adding to R a counter term $C(\Lambda, Q)$. We choose this function according to three criteria. First, the new function $\widetilde{R} \equiv R + C$ must again be a regulator. Second, we require that a zero is added for a particular value of Λ, say for $\Lambda = \mu$. Third, we require the first Λ-derivative of $\widetilde{R}$ to vanish at $\Lambda = \mu$. The conditions are met by

$$C(\Lambda, Q) = -Q^2 \frac{(\Lambda^2 - \mu^2)}{(\Lambda^2 + Q^2)^2}.$$

The kernel of the integral equation becomes then

$$U = -\frac{4}{3\pi^2} \frac{\alpha}{m} \left(\frac{2m^2}{Q^2} + R + C \right).$$

The lowest eigenvalue of the corresponding integral equation is displayed in Figure 2 as function of α and Λ.

Based on the Hellmann-Feynman theorem, and

$$\frac{d\widetilde{R}}{d\Lambda^2} = 2Q^2 \frac{(\Lambda^2 - \mu^2)}{(\Lambda^2 + Q^2)^3},$$

one expects that the derivative of the eigenvalues change sign at $\Lambda = \mu$. The numerical results in Figure 2 illustrate this very convincingly. In fact, for the numerical value $\mu = 1330$ MeV ($\mu/\Delta = 3.8$), the eigenvalues satisfies Eq.(1). The Hamiltonian is thus partially renormalized in the vicinity of $\Lambda \sim \mu$ for all α.

2. Exact renormalization by counter terms

Above, we have constructed a local renormalization counter term in the region of $\Lambda/\Delta = 3.8$. Now our aim is to renormalize globally, i.e. for all possible Λ. This can be achieved by requiring that the Λ^2-derivatives of all orders have to vanish in the point $\Lambda = \mu$. Besides that, we will take up an easier and more straightforward way to derive a global counter term.

The regularization function $\widetilde{R}$ is defined by:

$$\widetilde{R}(\Lambda, Q) = R(\Lambda, Q) + C(\Lambda, Q),$$
$$\text{with} \quad R(\Lambda, Q) = \frac{\Lambda^2}{\Lambda^2 + Q^2}.$$

Goal is to construct a counter term C such that

$$C(\Lambda = \mu, Q) = 0, \quad \text{and} \quad \left.\frac{d\widetilde{R}}{d\Lambda^2}\right|_{\forall\Lambda} = 0.$$

The requirements are satisfied by the differential equation

$$\frac{dC}{d\Lambda^2} = -\frac{dR}{d\Lambda^2} = -\frac{Q^2}{(\Lambda^2 + Q^2)^2}.$$

The boundary conditions are included by its integral form

$$\begin{aligned} C(\Lambda, Q) &= -\int_{\mu^2}^{\Lambda^2} d\lambda^2 \frac{dR(\lambda^2, Q)}{d\lambda^2} \\ &= \frac{\mu^2}{\mu^2 + Q^2} - \frac{\Lambda^2}{\Lambda^2 + Q^2}. \end{aligned}$$

The regularization function $\widetilde{R}$ becomes

$$\widetilde{R}(\Lambda, Q) = \frac{\mu^2}{\mu^2 + Q^2},$$

which is to be used in the integral equation of the $\uparrow\downarrow$-model, i.e.

$$\left[M^2 - 4m^2 - 4\vec{k}^2\right]\phi(\vec{k}) = \int d^3\vec{k}' \; U(\vec{k}', \vec{k})\phi(\vec{k}'),$$
$$\text{with} \quad U(\Lambda, Q) = -\frac{4}{3\pi^2}\frac{\alpha}{m}\left(\frac{2m^2}{Q^2} + \frac{\mu^2}{\mu^2 + Q^2}\right).$$

The equation is now manifestly independent of Λ and the limit $\Lambda \to \infty$ can be taken trivially. In line with the theory of renormalization, the three parameters α, μ and m have to be determined by experiment, i.e. in principal three exprimental values are needed to fix them.

3. Determining the parameters α and μ

We fix the two unknown parameters α and μ by the experimental values of the ground and excited state mass of the pion. The pion has the mass $M_\pi = 140$ MeV. The precise empirical value of the excited pion mass is not known very well. We choose here $M_{\pi^*} = M_\rho = 768$ MeV for no good reason other than convenience. This large value is the reason for our comparatively large quark mass $m = 406$ MeV, which is fixed here once and for all.

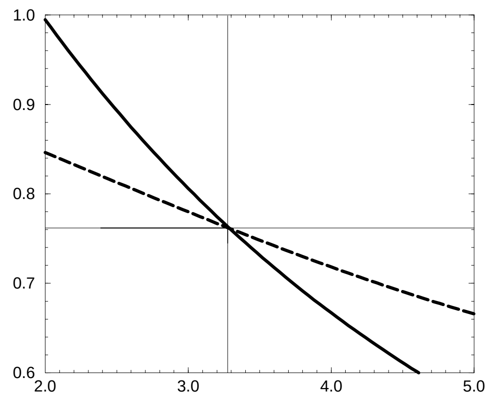

Figure 3. Two contours $\alpha_n(\mu)$ are plotted versus $2.0 \leq \mu/\Delta \leq 5.0$, with $\Delta = 350$ MeV. The solid contour $\alpha_\pi(\mu)$ is obtained by fixing the lowest eigenvalue to the pion ground state $M_0^2 = M_\pi^2 = (140\text{ MeV})^2$, while the dotted contour $\alpha_{\pi^*}(\mu)$ refers to the fixing of the second lowest eigenvalue to the first exited state of the pion $M_1^2 = M_{\pi^*}^2 = (768\text{ MeV})^2$.

Each of the two equations, $M_0^2(\alpha, \mu) = M_\pi^2$ and $M_1^2(\alpha, \mu) = M_{\pi^*}^2$ determine a function $\alpha(\mu)$, as illustrated in Figure 3. Their intersection point determines the solution, that is $\alpha_0 = 0.761$ and $\mu_0 = 1.15$ GeV, or $\mu_0/\Delta \sim 3.28$, as displayed in the figure.

Important to note is that the two contours in Figure 3 are intersecting only once. This crossing of the contours is unique, even for $\mu \to \infty$.

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REFERENCES

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